L^2-Betti Numbers of Algebras and Equivalence Relations

by

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Contents

Introduction

Atiyah [1] introduced the subject of L^2-invariants by defining L^2-Betti numbers for free actions of a countable group G on a manifold X such that $G\backslash X$ is compact. Later, Cheeger and Gromov [4] extended this definition to consider general group actions on CW-complexes, and through this defined L^2-Betti numbers of groups. An alternative path to defining these invariants was developed by Lück [13], and shown by him to be equivalent to that of Cheeger and Gromov.

Other objects have also had L^2-invariants associated to them. Gaboriau [11] introduced L^2-Betti numbers for equivalence relations in a manner inspired by the approach of Cheeger and Gromov, and used them to prove that the L^2-Betti numbers of orbit equivalent groups are identical. Sauer [17] reached the same result, defining L^2-Betti numbers of measurable groupoids in a manner inspired by the work of Lück. Since the equivalence relations considered by Gaboriau are measurable groupoids, Sauer's work provides an alternative definition of L^2-Betti numbers of equivalence relations, and the ways they reach their results show that the two definitions coincide in the case where the relation is induced by the free action of a group.

Finally, inspired by the work of Gaboriau and Lück, a definition of L^2-Betti numbers for von Neumann algebras was proposed by Connes and Shlyakhtenko [7].

In this thesis, we will present these various definitions of L^2-Betti numbers, with the main result being that the notions of L^2-Betti numbers of equivalence relations given by Gaboriau and Sauer coincide.

The thesis consists of the following parts:

Chapter 1 is a preliminary chapter, beginning with an auxiliary result on flatness of von Neumann algebras which extends Theorem 0.6(1) of [13]. Following this we state the main properties of the dimension function, following [13]. The third section follows [22] in introducing the rank metric and the completion functor, before the fourth section ties the two preceding sections together by considering the interplay between dimension and rank metric through results which are variations on those of [17, 22].

Chapter 2 considers the L^2-Betti numbers of groups, presenting the definitions of Cheeger and Gromov and Lück. An argument similar to that of

[13] shows that they agree. Some properties of these L^2-Betti numbers are also shown, mainly following [14].

Chapter 3 is the main part of the thesis, and considers L^2-Betti numbers of equivalence relations. After going through the necessary constructions, we show that the definitions of $\beta_n^{(2)}(R)$ given by Gaboriau and Sauer coincide, and in the process give an alternative proof of a result of Gaboriau.

Chapter 4 considers L^2-Betti numbers of von Neumann algebras, by first expanding on the motivation for the definition presented in [7] and then applying this to reach an alternative, relative definition of L^2-Betti numbers of von Neumann algebras which turns out to be equivalent to the original one through an excision-type result.

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Chapter 1

Preliminaries

1.1 Flatness of von Neumann algebras

Given an inclusion $N \subset M$ of von Neumann algebras, we want to show that M is flat as a right N-module. To do this, we first show that a von Neumann algebra M is semi-hereditary. Recall that a ring R is said to be *semihereditary* if any finitely generated ideal I of R is projective.

Proposition 1.1. *A von Neumann algebra is semihereditary.*

Proof. Let $I \subset M$ be a finitely generated left ideal. Then we can find a surjective map $T : M^n \to I$. Since we are considering left M-modules, this T is an element of $\mathrm{Mat}_{n\times 1}(A)$ acting on M^n by right multiplication.

Now, I is projective if it is a direct summand in M^n, and this is equivalent to $\ker T$ being a direct summand in M^n. However, $TT^* \in \mathrm{Mat}_n(M)$, and $\ker T = \ker TT^* = M^n(1 - r(|T|))$ where $r(|T|)$ is the range projection of $|T|$. Hence $I \simeq M^n r(|T|)$ is a direct summand, and hence projective. $\square$

Next, recall the following lemma, which is Exercise 2.26 p. 35 of [2] or, more explicitly, Proposition 2.3 of [3].

Lemma 1.2. *A right R-module A is flat if and only if for every finitely generated left ideal I of R the natural map*

$$A \otimes_R I \longrightarrow A \otimes_R R$$

is injective.

Proof. The only if part is obvious, so let us tackle the if part.

First, let $I \subset R$ be an ideal, and assume $\sum_{i=1}^n a_i \otimes x_i$ is in the kernel of the above map. Then the ideal $I_0 = \langle x_1, \dots, x_n \rangle \subset I$ is finitely generated, and $\sum a_i \otimes x_i \in A \otimes_R I_0$. Hence by assumption $\sum a_i \otimes x_i = 0$, so the map is injective. In particular, we see that B is A-flat for every cyclic R-module B.

Next, if $B = \langle b_1, \ldots, b_n \rangle$ is finitely generated, put $B_i = \langle b_1, \ldots, b_i \rangle$, and note that we have exact sequences

$$0 \longrightarrow B_{i-1} \longrightarrow B_i \longrightarrow B_i/B_{i-1} \longrightarrow 0$$

with B_i/B_{i-1} cyclic. Since in an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

with Z A-flat we have that Y is A-flat if X is A-flat, we can step up the ladder of exact sequences to conclude that B is A-flat.

Finally, let $\phi : B \to B'$ be a map of R-modules, and consider an element

$$\sum_{i=1}^{n} a_i \otimes b_i \in \ker(\mathrm{id} \otimes \phi : A \otimes_R B \to A \otimes_R B').$$

Then the submodules $B_0 = \langle b_1, \ldots, b_n \rangle$ and $B'_0 = \langle \phi(b_1), \ldots, \phi(b_n) \rangle$ of B and B' respectively are finitely generated. Hence so is B'_0/B_0, so this module is A-flat. Thus $\mathrm{id} \otimes \phi$ is injective when restricted to $A \otimes_R B_0$, and so $\sum a_i \otimes b_i$ was equal to zero to begin with.

Hence A is flat. □

Proposition 1.3. *Let $N \subset M$ be von Neumann algebras with N containing the unit of M. Then M is flat over N.*

Proof. By the lemma, it suffices to show that $M \otimes_N I \to M$ is injective for all finitely generated left ideals I of N.

Now, let I be a finitely generated ideal in N. Then I is projective, and so I is a direct summand in N^n for some n, say $N^n \simeq I \oplus J$. Next,

$$M \otimes_N I \longrightarrow M \otimes_N N$$

is injective if

$$M \otimes_N (I \oplus J) \longrightarrow M \otimes_N (N \oplus N^n)$$

is injective. Hence it suffices to show that $\mathrm{id} \otimes T : M \otimes_N N^k \to M \otimes_N N^l$ is injective for all injective $T : N^k \to N^l$. Indeed, T is given by right multiplication by a matrix in $\mathrm{Mat}_{k \times l}(N)$, so the induced map $\mathrm{id} \otimes T$ is given by right multiplication by the same matrix. But the original T is injective if and only if $s(|T|) = 1$, so clearly then $\mathrm{id} \otimes T$ is injective as well. □

1.2 The dimension function

Let M be a finite von Neumann algebra with trace τ. (By trace, we will in the following mean a finite faithful normal tracial state.) For a finitely generated projective M-module P, we have a surjection $M^n \to P$, which splits to give an embedding of P into M^n as a direct summand. Hence $P \simeq M^n p$ for some projection $p \in \mathrm{Mat}_n(M)$.

We now want to define

$$\dim_M P = \tau(p) = \sum_{i=1}^{n} \tau(p_{ii}).$$

To see that this is well defined, assume $\phi : P \to M^n p$ and $\psi : P \to M^m q$ are isomorphisms. We may assume $n = m$, since we otherwise may extend p, say, by zeroes. Then define $u, v \in \mathrm{Hom}_M(M^n, M^n)$ by putting

$$u(x) = \begin{cases} \psi(\phi^{-1}(x)) & \text{for } x \in M^n p \\ 0 & \text{otherwise} \end{cases}$$

and similarly

$$v(x) = \begin{cases} \phi(\psi^{-1}(x)) & \text{for } x \in M^n q \\ 0 & \text{otherwise.} \end{cases}$$

Then $u, v \in \mathrm{Mat}_n(M)$, and we have $p = vu$, $q = uv$, and hence

$$\tau(p) = \tau(uv) = \tau(vu) = \tau(q),$$

so $\dim_M P$ is well defined.

To extend the dimension function to all M-modules, we follow [13].

Definition 1.4. For an M-module A, define its *(von Neumann) dimension* to be

$$\dim_M A = \sup\{\dim_M P : P \subset A \text{ is finitely generated projective}\}.$$

The following is the main theorem of [13].

Theorem 1.5 (Lück).

(i) If P is a finitely generated projective M-module, the two definitions above coincide;

(ii) If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of M-modules, then

$$\dim_M B = \dim_M A + \dim_M C;$$

(iii) If A_i is a cofinal system of submodules of A, that is, if $A = \bigcup_{i \in I} A_i$ and for any two $i, j \in I$ there is $k \in I$ with $A_i, A_j \subset A_k$, then

$$\dim_M A = \sup_{i \in I} \dim_M A_i;$$

(iv) If A_i is a directed system of M-modules, and there for every i is a $j \geq i$ such that $\dim_M \mathrm{im}(A_i \to A_j) < \infty$, then

$$\dim \mathrm{colim}_i A_i = \sup_i \inf_{j \geq i} \dim_M \mathrm{im}(A_i \to A_j).$$

Definition 1.6. Given a map $\phi : A \to B$ of M-modules, we say that ϕ is a $\dim_M$-isomorphism if $\dim_M \ker \phi = \dim_M \operatorname{coker} \phi = 0$.

We will also need the following result, which consists of parts of Theorem 6.7 and Lemma 6.28 of [14].

Lemma 1.7. *Let A be a finitely presented M-module. Then there is a splitting of A as $A = PA \oplus TA$ where PA is finitely generated projective and there is a short exact sequence*

$$0 \longrightarrow M^n \longrightarrow M^n \longrightarrow TA \longrightarrow 0.$$

Lemma 1.8 (Sauer [17]). *Let M and N be von Neumann algebras, and let F : N-mod$\to$ M-mod be an exact functor which preserves colimits. If there is a constant $C > 0$ such that*

$$\dim_M F(P) = C \cdot \dim_N P$$

for every finitely generated projective N-module P, then

$$\dim_M F(A) = C \cdot \dim_N A$$

for every N-module A.

Proof. First assume A is finitely presented. Then by the preceding lemma we have a splitting of A as $PA \oplus TA$ with PA finitely generated projective. The module TA has an exact resolution

$$0 \longrightarrow N^n \longrightarrow N^n \longrightarrow TA \longrightarrow 0,$$

so additivity gives $\dim_N TA = 0$, and applying the exact functor F to this resolution, we again get $\dim_M F(TA) = 0$. Since we know from the assumption that $\dim_M F(PA) = C \cdot \dim_N PA$, we get

$$\dim_M F(A) = \dim_M F(PA) \oplus F(TA) = C \cdot \dim_N PA = \dim_N A.$$

Next, assume A is finitely generated. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow 0$$

of M-modules with $P \simeq M^n$ for some n. The kernel K is the directed union of its finitely generated submodules, $K = \bigcup_{i \in I} K_i$. Hence we get

$$\dim_N A = \dim_N P - \dim_N K = \dim_N P - \sup_{i \in I} \dim_N K_i = \inf_{i \in I} \dim_N P/K_i.$$

Now $F(K) = \operatorname{colim}_{i \in I} F(K_i)$ since F is exact and preserves colimits. Hence we get similarly

$$\dim_M F(A) = \inf_{i \in I} \dim_M F(P/K_i),$$

and since the P/K_i are finitely presented, we get $\dim_M F(A) = C \cdot \dim_N A$ for A finitely generated.

Finally, for an arbitrary module A, we need simply recall that A is the directed union of its finitely generated submodules. Hence the cofinality property of the dimension function reduces this to the finitely generated case. $\square$

In particular, if $N \subset M$ are von Neumann algebras, then the functor $M \otimes_N -$ satisfies the assumptions of the lemma.

Theorem 1.9 (Sauer [17]). *Let $N \subset M$ be von Neumann algebras, and let A be an N-module. Then*

$$\dim_N A = \dim_M M \otimes_N A.$$

Proof. By the lemma, it suffices to show this for finitely generated projective modules A. However, if $A \simeq N^n p$ for some $p \in \mathrm{Mat}_n(N)$, then $M \otimes_N A \simeq M^n p$, and $\dim_M M \otimes_N A = \tau(p) = \dim_N A$. $\square$

1.3 Rank metric and completion

Let M be a von Neumann algebra with trace τ, and let A be an M-module. For $\xi \in A$, define the *rank* of ξ to be

$$[\xi]_M = \inf\{\tau(p) : p \in P_M, p\xi = \xi\}$$

where P_M is the set of projections in M. When the von Neumann algebra M is understood, we simply write $[x]$ for $[x]_M$. This will be the case for the rest of this section.

Remark. This definition is not the same as that presented by Thom in [22]. There, he considers an M-bimodule A and defines

$$[x] = \inf\{\tau(p) + \tau(q) : p, q \in P_M, pxq = x\}.$$

This definition works equally well if we rather than M-bimodules consider M-N-bimodules. Our situation then corresponds to the case when N consists of the scalars.

In the simplest case, calculating the rank of an element is simple.

Example 1.10. Consider M as a module over itself. Then $[x] = \tau(r(x))$ for all $x \in M$. Indeed, if $px = x$ then $r(x) \leq p$, so $\tau(r(x)) \leq [x]$. But the other inequality is obvious.

We wish to use the rank of elements to introduce a notion of distance in A. We cannot do this perfectly. That is, there may be elements $\xi \in A$ with $[\xi] = 0$. Hence, the rank does not induce a metric on A. However, we do have the next best thing.

Lemma 1.11. *The function $d(\xi, \eta) = [\xi - \eta]$ defines a quasi-metric on A.*

Proof. It suffices to show that $[\xi + \eta] \leq [\xi] + [\eta]$. To this end, fix $\epsilon > 0$ and find projections p, q in M such that $p\xi = \xi$, $q\eta = \eta$ and $\tau(p) \leq [\xi] + \epsilon$, $\tau(q) \leq [\eta] + \epsilon$. Then $(p \vee q)(\xi + \eta) = \xi + \eta$, and we get

$$[\xi + \eta] \leq \tau(p \vee q) \leq \tau(p) + \tau(q) \leq [\xi] + [\eta] + 2\epsilon,$$

and since this holds for all ϵ, we get $[\xi + \eta] \leq [\xi] + [\eta]$. $\square$

Lemma 1.12. *Let $\phi : A \to B$ be a homomorphism of M-modules. Then*
(i) ϕ is a contraction in rank metric;
(ii) if ϕ is surjective, $\epsilon > 0$ and $\eta \in B$, there is a $\xi \in A$ with $\phi(\xi) = \eta$ and $[\xi] \leq [\eta] + \epsilon$.

Proof. For part (i), let $\xi \in A$ and let $p \in P_M$ satisfy $p\xi = \xi$. Then we also have $p\phi(\xi) = \phi(p\xi) = \phi(\xi)$, so that $[\phi(\xi)] \leq [\xi]$.

For part (ii), let $p \in P_M$ satisfy $p\eta = \eta$ and $[\eta] \leq \tau(p) + \epsilon$. Next, let ξ' be an arbitrary element in A with $\phi(\xi') = \eta$, and let $\xi = p\xi'$. We then have $\phi(\xi) = \phi(p\xi') = p\phi(\xi') = p\eta = \eta$, while $p\xi = \xi$, so $[\xi] \leq \tau(p) \leq [\eta] + \epsilon$. $\square$

Lemma 1.13. *For $x \in M$, the map $\lambda_x : A \to A$ of left multiplication by x is a contraction in rank metric. In particular, the set $N = \{x \in A : [x] = 0\}$ is a submodule of A.*

Proof. Let $\xi \in A$, let $\epsilon > 0$, and let $p \in P_M$ satisfy $px = x$ and $\tau(p) \leq [\xi] + \epsilon$. Then $[\lambda_x \xi] = [x\xi] = [xp\xi]$. Hence we get that $r(xp)xp\xi = xp\xi = x\xi$, and so $[\lambda_x \xi] \leq \tau(r(xp))$. However, we have $\tau(r(xp)) = \tau(s(xp)) \leq \tau(p)$, so that $[\lambda_x \xi] \leq [\xi]$. $\square$

We can now do what we usually do given a quasi-metric on a set, namely complete it.

Definition 1.14. Let A be an M-module, and let $N \subset A$ be the submodule

$$N = \{x \in A : [x] = 0\}.$$

We then write $c_M(A)$ for the completion of A/N with respect to the rank metric induced by M.

Note that we have a natural map $A \to c(A)$ with kernel N. If this map is an isomorphism, we say that A is *complete*. If $B \subset A$ satisfies $c(B) = c(A)$ we say that B is *M-dense* in A.

Lemma 1.15. *Completion is a functor from the category of M-modules to itself.*

Proof. Since $\lambda_x : A \to A$ is a contraction in rank metric, it extends to the completions. The fact that c is a functor follows from the fact that any homomorphism $\phi : A \to B$ is a contraction in rank metric. $\square$

Proposition 1.16. *The completion functor is exact.*

Proof. Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

be an exact sequence of M-modules. We must show that

$$0 \longrightarrow c(A) \longrightarrow c(B) \longrightarrow c(C) \longrightarrow 0$$

is exact.

For exactness at $c(C)$, let ζ_n be a Cauchy sequence in C. We may assume that $[\zeta_n - \zeta_{n+1}] \leq 2^{-n}$. We can then lift ζ_n to a sequence η_n in B with $[\eta_n - \eta_{n+1}] \leq 2^{1-n}$. Hence η_n is a Cauchy sequence mapping to ζ_n, so $c(\pi)$ is surjective.

Next, consider the case of $c(B)$. Clearly $\operatorname{im} c(i) \subset \ker c(\pi)$, so we must show the opposite inclusion. To this end, let η_n be a Cauchy sequence in B which maps to zero in $c(C)$. Then $[\pi(\eta_n)]$ tends to zero, so we can lift $\pi(\eta_n)$ to a Cauchy sequence η'_n in B such that $[\eta'_n]$ tends to zero. Now $(\eta_n - \eta'_n)$ is again Cauchy in B, and equivalent to the sequence η_n. However, $(\eta_n - \eta'_n)$ is actually contained in $\operatorname{im} i$ by the exactness of the original sequence, so $\ker c(\pi) \subset \operatorname{im} c(i)$.

The exactness at $c(A)$ is clear, since the inclusion $i : A \to B$ is an isometry in rank metric. $\square$

Let us now consider M as a module over itself, and find its closure in the rank metric. To this end, recall from [19, IX.2] that the *measure topology* on M is given by the neighborhood system

$$N(\epsilon, \delta) = \{a \in M : \exists p \in P_M, \|ap\| < \epsilon, \tau(p^\perp) < \delta\}$$

for $\epsilon, \delta > 0$, and translates of this system. By [19, Corollary IX.2.9], the completion of M in this topology is exactly the algebra of affiliated operators of M, denoted by $\mathcal{U}(M)$. The following is Lemma IX.2.9(ii) of [19].

Lemma 1.17. *For each $a \in \mathcal{U}(M)$ and $\epsilon > 0$ there is $p \in P_M$ with $\tau(p^\perp) < \epsilon$ and $ap \in M$.*

Proof. Let $a \in \mathcal{U}(M)$, and let a_n be a sequence in M converging to a in the measure topology. We may then assume that

$$a = a_1 + \sum_{i=1}^{\infty} (a_{k+1} - a_k), \qquad a_{k+1} - a_k \in N(2^{-k}, 2^{-k}).$$

Setting $b_k = a_{k+1} - a_k$, we may choose $q_k \in P_M$ such that $\|b_k q_k\| < 2^{-k}$ and $\tau(q_k^\perp) < 2^{-k}$. Let $p_n = \bigwedge_{k \geq n} q_k$. Then the sequence p_n is increasing, and $\tau(p_n^\perp) < 2^{-n+1}$. Hence the p_n converge to 1. We get that

$$ap_n = a_1 p_n + \sum_{k=1}^{\infty} b_k p_n = a_1 p_n + \sum_{k=1}^{n-1} b_k p_n + \sum_{k=n}^{\infty} b_k q_k p_n.$$

The last sum converges in norm as $\|b_k q_k\| < 2^{-k}$, so this is an element of M. $\square$

Corollary 1.18. *M is M-dense in $\mathcal{U}(M)$.*

Proof. Since the measure topology is weaker that the topology induced by the rank metric, we clearly have $c_M(M) \subset \mathcal{U}(M)$. To see that we have equality, let $a \in \mathcal{U}(M)$ and $\epsilon > 0$. By the lemma there is $p \in P_M$ with $\tau(p^\perp) < \epsilon$ and $a^*p \in M$. Hence $pa \in M$, and $[pa - a] < \tau(p^\perp) < \epsilon$, so M is M-dense in $\mathcal{U}(M)$. $\square$

Corollary 1.19. *M is M-dense in $L^2(M)$.*

Proof. By [19, Theorem IX.2.13] we may identify $L^2(M)$ with a subset of $\mathcal{U}(M)$. Hence the result follows from the above corollary. $\square$

1.4 Completion and dimension

There is a close connection between the dimension function and the completion functor, which is implicit in [22]. The basis of this connection consists of the following results. The first is the so-called "local criterion" of [17], reformulated to our setting.

Theorem 1.20 (Sauer). *For an M-module A, we have $\dim_M A = 0$ if and only if $c_M(A) = 0$.*

Proof. First, assume $\dim_M A = 0$, and choose $a \in A$ and $\epsilon > 0$. We want to show that $[a]_M < \epsilon$. To this end, consider the map

$$\phi : M \to A, \qquad \phi(x) = xa.$$

Then $\dim_M \ker \phi = 1$, and by [14, Exercise 6.3], there is a submodule $P \subset \ker \phi$ which is a direct summand in M and satisfies $\dim_M P > 1 - \epsilon$. That is, $P = M(1-p)$ for some $p \in P_M$ with $\tau(p) < \epsilon$. Since $(1-p) \in \ker \phi$, we get that $pa = a$, so $[a]_M \le \tau(p) < \epsilon$, as desired. Hence $c_M(A) = 0$.

For the converse, suppose $\dim_M A > 0$, so there is a nontrivial finitely generated projective submodule $P \subset A$. Then there is a nontrivial M-homomorphism $\phi : P \to M$. Indeed, there is a surjection $M^n \to P$, and hence by projectiveness a map $P \to M^n$ which we may compose with the projection to a summand to get a nontrivial map. We may thus choose $x = \phi(a) \neq 0$ in the image of ϕ. However, given $\epsilon > 0$, there is then $p \in P_M$ with $\tau(p) < \epsilon$ and $pa = a$. Hence $px = p\phi(a) = \phi(pa) = \phi(a) = x$, so that $r(x) \le p$, or $\tau(r(x)) \le \tau(p) < \epsilon$. Since this holds for all $\epsilon > 0$, we must have $r(x) = 0$, and hence $x = 0$ which is a contradiction. $\square$

Note that this implies that $\phi : A \to B$ is a $\dim_M$-isomorphism if and only if $\ker \phi$ and $\operatorname{coker} \phi$ have vanishing completions. That is, if $c(\phi) : A \to B$ is an isomorphism.

Corollary 1.21. *For all M-modules A, the canonical map $c : A \to c_M(A)$ is a $\dim_M$-isomorphism.*

Proof. Consider the exact sequence

$$0 \longrightarrow \ker c \longrightarrow A \longrightarrow c_M(A) \longrightarrow \operatorname{coker} c \longrightarrow 0.$$

After completion with respect to the rank metric, we see that the central map is an isomorphism, so that $c_M(\ker c) = c_M(\operatorname{coker} c) = 0$. Hence by the observation above, the map is a $\dim_M$-isomorphism. □

In particular, the inclusion of M into $L^2(M)$ is a dimension isomorphism.

We may now explore closer the relations mentioned by both Sauer and Thom between the dimension function and the localization of the category of M-modules by a certain Serre subcategory. To make this more concrete, however, let us first recall some terminology, following [16]. (There the terminology "dense subcategory" is used instead of "Serre subcategory".)

Hence, let $\mathcal{C}$ be an Abelian category, and let $\mathcal{A}$ be a full subcategory. Say that $\mathcal{A}$ is a *Serre subcategory* of $\mathcal{C}$ if for any exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{C}$, B is an object of $\mathcal{A}$ if and only if both A and C are objects of $\mathcal{A}$.

Example 1.22. The example we will care about is that where $\mathcal{C}$ is the category of M-modules, and $\mathcal{A}$ is the subcategory of zero-dimensional M-modules. Then $\mathcal{A}$ is a Serre subcategory, since dimension is additive over exact sequences.

The important result is then the following, which is Theorems 3.3 and 3.8 of [16].

Theorem 1.23. *Let $\mathcal{A}$ be a Serre subcategory of the Abelian category $\mathcal{C}$. Then there is an Abelian category $\mathcal{C}/\mathcal{A}$ with the same objects as $\mathcal{C}$, in which $\pi : A \to B$ is an isomorphism if and only if $\ker \phi$ and $\operatorname{coker} \phi$ are elements of $\mathcal{A}$. Furthermore, the functor $\mathcal{C} \to \mathcal{C}/\mathcal{A}$ is exact.*

We will not consider the construction implicit in the above theorem, since the completion functor does exactly this. Indeed, the core idea of [22] is that the functor $\mathcal{C} \to \mathcal{C}/\mathcal{A}$ can be identified with the completion functor (after possibly identifying some isomorphic objects). Hence we have an Abelian category M-mod_c of complete M-modules, in which isomorphism is exactly dimension isomorphism.

Lemma 1.24. *(i) The completion functor is left adjoint to the functor embedding M-mod$_c$ into M-mod. That is*

$$\mathrm{Hom}_M(c_M(L), K) = \mathrm{Hom}_M(L, K)$$

for all complete M-modules K and M-modules L.

(ii) The completion functor preserves projective objects.

(iii) The category M-mod$_c$ is Abelian with enough projectives.

Proof. (i) If K is complete, the natural map $K \to c(K)$ is an isomorphism, so applying the functor c defines a natural map

$$\mathrm{Hom}_M(L, K) \longrightarrow \mathrm{Hom}_M(c(L), K).$$

A map in the inverse direction is given by precomposition with the map $L \to c(L)$. Now, the result follows from the properties of the rank metric, since $\mathrm{im}(L)$ is rank dense in $c(L)$, and since $\{\xi \in K : [\xi] = 0\} = \{0\}$.

(ii) Recall that P is projective if and only if the functor $X \mapsto \mathrm{Hom}(P, X)$ is exact. But that this implies the same for $c(P)$ is immediate from (i).

(iii) M-mod$_c$ is Abelian since completion is exact. Indeed, we can form kernels and cokernels in M-mod, and transport them over through completion. By part (ii), the same can be done to projective resolutions. $\square$

We will in the following need some results on how the Tor functors behave with respect to dimension isomorphisms.

Proposition 1.25. *Let R be a ring, and let $\phi : A_1 \to A_2$ be a map of M-R-modules which is a $\dim_M$-isomorphism. Then for all R-modules B, the induced maps*

$$\mathrm{Tor}_n^R(A_1, B) \longrightarrow \mathrm{Tor}_n^R(A_2, B)$$

are $\dim_M$-isomorphisms.

Proof. Let A be an M-R-module with $\dim_M A = 0$, let B be an arbitrary R-module, and let $P_* \to B$ be a free resolution of B. Then $\dim_M(A \otimes_R P_n) = 0$ for all n by additivity of the dimension, and so we have

$$\dim_M \mathrm{Tor}_n^R(A, B) = \dim_M H_n(A \otimes_R P_*) = 0.$$

In the case of a general $\dim_M$-isomorphism $\phi : A_1 \to A_2$, consider the short exact sequences

$$0 \longrightarrow \ker\phi \longrightarrow A_1 \longrightarrow \mathrm{im}\,\phi \longrightarrow 0$$

$$0 \longrightarrow \mathrm{im}\,\phi \longrightarrow A_2 \longrightarrow \mathrm{coker}\,\phi \longrightarrow 0.$$

We get long exact sequences in Tor, and since we know that

$$\dim_M \ker \phi = \dim_M \operatorname{coker} \phi = 0$$

the first part of the proof shows that the corresponding Tor-terms are zero-dimensional. What remains tells us that the maps

$$\operatorname{Tor}_n^R(A_1, B) \longrightarrow \operatorname{Tor}_n^R(\operatorname{im} \phi, B) \longrightarrow \operatorname{Tor}_n^R(A_2, B)$$

are $\dim_M$-isomorphisms, and hence so is their composition. $\square$

The following is Lemma 1.1 of [22].

Lemma 1.26. *Let $\mathcal{A}$ be an Abelian category with enough projectives, let $F, G : \mathcal{A} \to M$-mod be right exact functors, and assume there exists a natural transformation $h : F \to G$ such that $c_M(h)$ consists of isomorphisms. Then the induced natural transformations $h_i : L_iF \to L_iG$ also have $c_M(h_i)$ consisting of isomorphisms.*

Proof. Let $P_* \to A$ be a projective resolution of $A \in \mathcal{A}$. Then the left derived functors of F and G are given by $H_n(F(P_*))$ and $H_n(G(P_*))$ respectively. However, if we move to completions, we have that

$$c(H_n(F(P_*))) = H_n(c(F(P_*))) \simeq H_n(c(G(P_*))) = c(H_n(G(P_*)))$$

which was what we wanted. $\square$

Recall that a mapping $T : X \to X$ on a metric space X is said to be *Lipschitz* if there is a constant C such that $d(Tx_1, Tx_2) \leq Cd(x_1, x_2)$. The importance of this concept for our purposes is that it is a sufficiently weak condition on T, but still guarantees that T can be extended by continuity to the completion of X with respect to the metric.

Lemma 1.27. *Let $N \subset R \subset M$ be rings, with N, M von Neumann algebras. Let B be an R-module, and assume R acts on B as Lipschitz operators with respect to the rank metric induced by N. Then*

$$M \otimes_R B \longrightarrow M \otimes_R c_N(B)$$

is a $\dim_M$-isomorphism.

Proof. Consider the map $\phi : M \otimes_R c_N(B) \to c_M(M \otimes_R B)$. This is well defined, since if $m_1, \ldots, m_n \in M$ and $\{\xi_{ik}\}_{k=1}^{\infty}$ are Cauchy sequences in B with respect to the N-rank metric, then

$$x_k = \sum_{i=1}^{n} m_i \otimes \xi_{ik}$$

is a Cauchy sequence in $M \otimes_R B$ with respect to the M-rank metric. Indeed, if $\xi \in B$ with $p\xi = \xi$ then

$$[m \otimes \xi]_M = [m \otimes p\xi]_M = [mp \otimes \xi]_M \leq [mp]_M \leq \tau(p),$$

and since this holds for all such p, $[m \otimes \xi]_M \leq [\xi]_N$, and x_k is Cauchy. Hence ϕ is well defined and is a contraction. Thus ϕ extends to a map

$$\phi : c_M(M \otimes_R c_N(B)) \to c_M(M \otimes_R B).$$

On the other hand, we have the natural map

$$\psi : c_M(M \otimes_R B) \to c_M(M \otimes_R c_N(B)).$$

This is clearly well defined, and is again a contraction. Now $\phi \circ \psi$ is the identity on the dense subspace $M \otimes_R B$ of $c_M(M \otimes_R B)$. Hence it is the identity on all of $c_M(M \otimes_R B)$, and likewise for $\psi \circ \phi$ on $c_M(M \otimes_R c_N(B))$. Thus ϕ and ψ are isomorphisms, and so the original map is a $\dim_M$-isomorphism. $\square$

It follows from this that if B_1 and B_2 are R-modules on which R acts as Lipschitz operators, and $h : B_1 \to B_2$ is a $\dim_N$-isomorphism, then

$$\mathrm{id} \otimes h : M \otimes_R B_1 \to M \otimes_R B_2$$

is a $\dim_M$-isomorphism.

For the next result, we will need some bits of homological algebra, or more exactly the Grothendieck spectral sequence. Before stating the needed result, let us recall some basics, following Sections 5.2 and 5.8 of [23]. (Note that the following is a little bit more general than what we really need.)

Definition 1.28. A *first quadrant homology spectral sequence starting with* E^a in an Abelian category $\mathcal{A}$ consists of the following data:

(i) A family $\{E^r_{pq}\}$ of objects of $\mathcal{A}$ defined for integers $p, q \geq 0$ and $r \geq a$;
(ii) Maps $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ that satisfy $d^r \circ d^r = 0$;
(iii) Isomorphisms $E^{r+1}_{pq} \simeq \ker d^r_{pq} / \operatorname{im} d^r_{p+r,q-r+1}$.

Note that there for each p and q is some r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for $r \geq r_0$. Write E^∞_{pq} for this stable value.

Say that a first quadrant homology spectral sequence *converges to* H_* if there is a family of objects H_n of $\mathcal{A}$, each with a finite filtration

$$0 = F_s H_n \subset \cdots \subset F_{p-1} H_n \subset F_p H_n \subset F_{p+1} H_n \subset \cdots \subset F_t H_n = H_n,$$

together with isomorphisms $E^\infty_{pq} \simeq F_p H_{p+q} / F_{p-1} H_{p+q}$. If this is the case, we write

$$E^a_{pq} \Rightarrow H_{p+q}.$$

Theorem 1.29 (Grothendieck spectral sequence). *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Suppose furthermore that we have right exact functors*

$$\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C}$$

such that $G(A)$ is F-acyclic for all projective objects $A \in \mathcal{A}$. Then there is a convergent first quadrant homology spectral sequence starting with E^2 such that

$$E^2_{pq} = (L_pF)(L_qG)(A) \Rightarrow L_{p+q}(FG)(A)$$

for each $A \in \mathcal{A}$.

Proposition 1.30. *Let $N \subset R \subset M$ be as above, and assume that for any R-module B R acts on B as Lipschitz operators with respect to the rank metric induced by N. Then for any $\dim_N$-isomorphism $\phi : B_1 \to B_2$, the induced maps*

$$\mathrm{Tor}_n^R(M, B_1) \longrightarrow \mathrm{Tor}_n^R(M, B_2)$$

are $\dim_M$-isomorphisms.

Proof. Since $c_N : N\text{-mod} \to N\text{-mod}_c$ is exact, we have a natural identification between the left derived functors

$$L_i(M \otimes_R -) \circ c = L_i(M \otimes_R c_N(-)).$$

Indeed, we can apply the Grothendieck spectral sequence with $\mathcal{A} = N$-mod, $\mathcal{B} = N\text{-mod}_c$ and $\mathcal{C} = M$-mod, $G = c_N$ and $F = M \otimes_R -$. Since c_N preserves projective objects, this satisfies the conditions of the theorem. Hence the exactness of c_N shows that for each $A \in R$-mod we have a convergent spectral sequence with E^2_{pq}-terms

$$E^2_{pq} = (L_pF)(L_qG)(A) = \begin{cases} \mathrm{Tor}_p^R(M, c_N(A)) & \text{for } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

converging to $L_{p+q}(FG)(A)$. Since the sequence collapses immediately, we have the indicated identification.

Now, the previous two lemmas implies the existence of a natural map

$$\mathrm{Tor}_n^R(M, -) \longrightarrow L_i(M \otimes_R c_N(-))$$

which is a $\dim_M$-isomorphism. Combining these observations, we see that

$$\dim_M \mathrm{Tor}_n^R(M, L) = 0$$

whenever $c_N(L) = 0$.

In the case of a general $\dim_N$-isomorphism $\phi : B_1 \to B_2$, consider the short exact sequences

$$0 \longrightarrow \ker\phi \longrightarrow B_1 \longrightarrow \operatorname{im}\phi \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im}\phi \longrightarrow B_2 \longrightarrow \operatorname{coker}\phi \longrightarrow 0.$$

We get long exact sequences in Tor, and since both $\ker\phi$ and $\operatorname{coker}\phi$ are zero-dimensional, we know from the first part of the proof that the corresponding Tor-terms are zero-dimensional. What remains tells us that the maps

$$\operatorname{Tor}_n^R(M, B_1) \longrightarrow \operatorname{Tor}_n^R(M, \operatorname{im}\phi) \longrightarrow \operatorname{Tor}_n^R(M, B_2)$$

are $\dim_M$-isomorphisms, and hence so is their composition. □

Finally, let us note that when we are calculating the dimensions of left derived functors of a functor F, we can relax the demands on our resolutions. The proof is essentially the same as the usual proof that an F-acyclic resolution suffices.

Lemma 1.31. *Let F be a right exact functor from an Abelian category $\mathcal{A}$ with enough projectives into the category of M-modules. Then*

$$\dim_M L_iF(A) = \dim_M H_i(F(P_*))$$

for any resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

of A by objects P_k satisfying $\dim_M L_iF(P_k) = 0$ *for $i \geq 1$.*

Proof. First, let

$$0 \longrightarrow K \longrightarrow P \longrightarrow A \longrightarrow 0$$

be exact, with P satisfying $\dim_M L_iF(P) = 0$ for $i \geq 1$. Then

$$\dim_M L_iF(A) = \begin{cases} \dim_M L_{i-1}F(K) & \text{if } i \geq 2 \\ \dim_M \ker(F(K) \to F(P)) & \text{if } i = 1 \end{cases}$$

Indeed, we have a long exact sequence

$$\cdots \longrightarrow L_{i+1}F(P) \longrightarrow L_{i+1}F(A) \longrightarrow L_iF(K) \longrightarrow L_iF(P) \longrightarrow \cdots$$

and the result follows by passing to completions with respect to the rank metric with respect to M at every term.

Next, let $P_* \to A$ be a resolution of A with $\dim_M L_i F(P_k) = 0$ for $i \geq 1$, and consider the exact sequence

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

Then

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow K \longrightarrow 0$$

is a similar resolution of K.

Since F is right exact, we have

$$\dim_M L_0 F(A) = \dim_M F(A) = \dim_M H_0(F(P_*)).$$

Furthermore, we have

$$\begin{aligned}\dim_M L_1 F(A) &= \dim_M \ker(F(K) \to F(P_0)) \\ &= \dim_M \ker(F(P_1)/\operatorname{im} F(P_2) \to F(P_1)) \\ &= \dim_M H_1(F(P_*)).\end{aligned}$$

The general result now follows by induction, as we for $i \geq 2$ have

$$\begin{aligned}\dim_M L_i F(A) &= \dim_M L_{i-1} F(K) \\ &= \dim_M H_{i-1}(F(P_{*-1})) \\ &= \dim_M H_i(F(P_*)).\end{aligned}$$

□

Finally, let us note the following relation between the rank metric structure and Hilbert module structures.

Lemma 1.32. *Let $T : K \to H$ be a bounded map of Hilbert M-modules such that $\dim_M T(K) < \infty$. Then $T(K)$ is M-dense in $\overline{T(K)}$, the Hilbert space closure of $T(K)$.*

Proof. Consider $TT^* : \overline{T(K)} \to T(K)$. This is injective, since $\ker T^* = (\operatorname{im} T)^\perp$ whence T^* is injective on $\overline{T(K)}$. Hence $\dim_M T(K) = \dim_M \overline{T(K)}$, which implies the density, since the dimensions are finite. □

Chapter 2

L^2-Betti numbers of groups

Let G be a discrete countable group, and let X be a G-CW-complex. That is, X is a CW-complex on which G acts cellularly. Say that X is *free* if the G-action is free, and that X is *finite* if $G\backslash X$ is a finite CW-complex.

The main example to keep in mind in the following is that of a space X which is the universal covering space of a finite CW-complex space Y with $\pi_1(Y) = G$. For instance, consider the case $Y = \bigvee_{i=1}^n S^1$ and $G = \mathbb{F}_n$, the free group on n generators.

If X is a free finite G-CW-complex, we may choose one element for each G-orbit in $X^{(n)}$, say $x_1, \ldots, x_m$. We then have an isomorphism

$$C_n(X) \simeq \bigoplus_{i=1}^m \mathbb{Z}Gx_i$$

and may define the cellular complex of L^2-chains of X by

$$C_n^{(2)}(X) = \ell^2(G) \otimes_{\mathbb{Z}G} C_*(X) \simeq \bigoplus_{i=1}^m \ell^2(G)$$

where the boundary maps are induced by those of $C_*(X)$. Hence, we define the reduced L^2-homology and L^2-Betti numbers of X by

$$\begin{aligned} H_n^{(2)}(X; LG) &= \ker \partial_n / \overline{\operatorname{im} \partial_{n+1}}, \\ \beta_n^{(2)}(X; LG) &= \dim_{LG} H_n^{(2)}(X; LG). \end{aligned}$$

In the case mentioned above, where $X = EG$ for a group G and the complex $BG = G\backslash EG$ is finite, we define the n-th L^2-Betti number of the group G to be

$$\beta_n^{(2)}(G) = \beta_n^{(2)}(EG; LG).$$

Example 2.1 (L^2-Betti numbers of free groups). Consider the free group $\mathbb{F}_n = \langle a_1, \ldots, a_n \rangle$ acting on the tree T which is the universal covering space of $\bigvee_{i=1}^{n} S^1$. Choose a vertex x and edges e_i corresponding to the circles. Then

$$C_0(T,\mathbb{Z}) = \mathbb{Z}\mathbb{F}_n x, \qquad C_1(T,\mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z}\mathbb{F}_n e_i,$$

and we have boundary maps $\partial_1 e_i = (a_i - e)x$. On the ℓ^2-level, we get a chain complex

$$0 \longleftarrow \ell^2(\mathbb{F}_n) \xleftarrow{\partial_1} \bigoplus_{i=1}^{n} \ell^2(\mathbb{F}_n) \longleftarrow 0.$$

We claim that $\operatorname{im} \partial_1$ is M-dense in $\ell^2(\mathbb{F}_n)$. Indeed, let $s \in \ell^2(\mathbb{F}_n)$, and assume $\partial_1 s e_1 = 0$. Then, for all $g \in \mathbb{F}_n$ we have

$$0 = (\partial_1 s e_1)(g) = s(a_1 g) - s(g)$$

so $s(a_1 g) = s(g)$. Hence s must be constant on each set $\{a_1^n g\}_{n\in\mathbb{Z}}$, and hence zero since these sets are infinite. Thus $s = 0$, and ∂_1 is injective when restricted to $\ell^2(\mathbb{F}_n)e_1$.

It follows that $\dim_M \operatorname{im} \partial_1 \geq \dim_M \ell^2(\mathbb{F}_n) = 1$, but since $\operatorname{im} \partial_1 \subset \ell^2(\mathbb{F}_n)$ we must have $\dim_M \operatorname{im} \partial_1 = 1$, whence the image is M-dense.

Now we have short exact sequences

$$0 \longrightarrow \overline{\operatorname{im} \partial_1} \longrightarrow \ell^2(\mathbb{F}_n) \longrightarrow H_0^{(2)}(T,\mathbb{F}_n) \longrightarrow 0$$

$$0 \longrightarrow \ker \partial_1 \longrightarrow \bigoplus_{i=1}^{n} \ell^2(\mathbb{F}_n) \longrightarrow \operatorname{im} \partial_1 \longrightarrow 0,$$

and since $H_1^{(2)}(T,\mathbb{F}_n) = \ker \partial_1$ and by additivity of dimension, we get

$$\dim_M H_0^{(2)}(T,\mathbb{F}_n) = 0, \qquad \dim_M H_1^{(2)}(T,\mathbb{F}_n) = n-1.$$

2.1 Non-cocompact actions

Note that X is a finite G-CW-complex if and only if the action of G on X is cocompact. That is, if $G\backslash X$ is compact. The need to generalize the above definition is clear from the fact that not all groups G have a classifying space EG such that $G\backslash EG$ is compact. This is, for instance, the case if G contains an element of finite order. (See [12, Prop. 2.45].)

On the other hand, it is clear that the above approach does not work if X is not finite. Indeed, then the ∂_n are in general not bounded, so they are not defined as operators from $C_n^{(2)}(X) \to C_{n-1}^{(2)}(X)$.

Two ways to move around this problem have been proposed. First by Cheeger and Gromov [4], and later by Lück [13], who also showed that the two definitions coincide. Since the move to Lück's definition is perhaps the simpler of the two, we will consider it first.

Definition 2.2. Let X be a G-space. We then define the L^2-homology and L^2-Betti numbers of X with respect to G by

$$H_n^{(2)}(X; LG) = H_n(LG \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X)),$$
$$\beta_n^{(2)}(X; LG) = \dim_{LG} H_n^{(2)}(X; LG),$$

where $C_*^{\text{sing}}(X)$ is the singular chain complex of X. In particular, if G is a group, we define

$$\beta_n^{(2)}(G) = \beta_n^{(2)}(EG; LG).$$

To see that this coincides with our previous definition in the case where X is finite we simply combine the following two lemmas:

Lemma 2.3 ([13] Lemma 4.2). *Let X be a G-CW-complex. Then there is a $\mathbb{Z}G$-chain homotopy equivalence*

$$f_*(X) : C_*(X) \to C_*^{\text{sing}}(X)$$

which is unique up to $\mathbb{Z}G$-homotopy, and natural in X. In particular,

$$H_n(V \otimes_{\mathbb{Z}G} C_*(X)) \simeq H_n(V \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(X))$$

for any right $\mathbb{Z}G$-module V.

Lemma 2.4. *Let X be a free finite G-CW-complex. Then*

$$\dim_{LG} H_n^{(2)}(C_*^{(2)}(X)) = \dim_{LG} H_n(LG \otimes_{\mathbb{Z}G} C_*(X)).$$

Proof. Consider $H_n^{(2)}(C_*^{(2)}(X))$. We have a short exact sequence

$$0 \longrightarrow \overline{\operatorname{im} \partial_{n+1}} \longrightarrow \ker \partial_n \longrightarrow H_n^{(2)}(C_*^{(2)}(X)) \longrightarrow 0,$$

which gives

$$\begin{aligned}\dim_{LG} H_n^{(2)}(C_*^{(2)}(X)) &= \dim_{LG} \ker \partial_n - \dim_{LG} \overline{\operatorname{im} \partial_{n+1}} \\ &= \dim_{LG} \ker \partial_n - \dim_{LG} \operatorname{im} \partial_{n+1} \\ &= \dim_{LG} H_n(C_*^{(2)}(X)).\end{aligned}$$

Finally we have that

$$\dim_{LG} H_n(C_*^{(2)}) = \dim_{LG} H_n(LG \otimes_{ZG} C_*(X))$$

since we have a sequence of $\dim_{LG}$-isomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & LG \otimes_{\mathbb{Z}G} C_n(X) \arrow[r] \arrow[d] & LG \otimes_{\mathbb{Z}G} C_{n-1}(X) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \ell^2(G) \otimes_{\mathbb{Z}G} C_n(X) \arrow[r] & \ell^2(G) \otimes_{\mathbb{Z}G} C_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

which become isomorphisms after completion with respect to the rank metric induced by the LG-structure.

□

It is also convenient to note here another expression for the L^2-Betti numbers of a group G. First, recall that if X is a contractible space on which G acts freely, for instance $X = EG$, then

$$0 \longleftarrow \mathbb{C} \longleftarrow C_0(X;\mathbb{C}) \longleftarrow C_1(X;\mathbb{C}) \longleftarrow \cdots$$

is a free resolution of $\mathbb{C}$ as a $\mathbb{C}G$-module. Hence, we see that

$$\begin{aligned}\beta_n^{(2)}(G) &= \dim_{LG} H_n(LG \otimes_{\mathbb{Z}G} C_*(EG)) = \dim_{LG} H_n(LG \otimes_{\mathbb{C}G} C_*(EG;\mathbb{C}) \\ &= \dim_{LG} \operatorname{Tor}_n^{\mathbb{C}G}(LG,\mathbb{C}).\end{aligned}$$

Let us now consider the definition given by Cheeger and Gromov [4]. First, however, we have to define L^2-cohomology of a space X.

Definition 2.5. Let X be a finite G-CW-complex. Then let $C^n_{(2)}(X)$ be the set of square-summable functions $f : X^n \to \mathbb{C}$, and define the coboundary operator $\partial^n : C^n_{(2)}(X) \to C^{n+1}_{(2)}(X)$ by $\partial^n f = f \circ \partial_{n+1}$. The (reduced) L^2-cohomology of X is then given by

$$H^n_{(2)}(X;LG) = \ker \partial^n / \overline{\operatorname{im} \partial^{n-1}}.$$

Note that there is as usual a clear duality between L^2-homology and L^2-cohomology. First, each $C^k_{(2)}(X)$ is isomorphic to $C_k^{(2)}(X)$ by

$$\phi(f) = \sum_{x \in X^k} f(x)x,$$

and under this identification $\partial^n = \partial^*_{n+1}$. Furthermore,

$$\begin{aligned}H_n^{(2)}(X;LG) &= \ker \partial_n / \overline{\operatorname{im} \partial_{n+1}} = \overline{\operatorname{im} \partial^{n-1}}^{\perp} / (\ker \partial^n)^{\perp} \\ &= \ker \partial^n / \overline{\operatorname{im} \partial^{n-1}} = H^n_{(2)}(X;LG)\end{aligned}$$

so that the L^2-Betti numbers defined by homology and cohomology coincide.

To return to the definition, let Y be a G-space, and let $\mathcal{C}(Y)$ be the category of pairs (X, f) where X is a free finite G-CW-complex and f is a G-equivariant map $f : X \to Y$. The set of morphisms between (X_1, f_1) and (X_2, f_2) is empty unless $X_1 \subset X_2$ and $f_2|_{X_1} = f_1$, in which case it contains a single element.

Definition 2.6. Let Y be a G-space. Then define the L^2-cohomology of Y by

$$H^k_{(2)}(Y;LG) = \lim_{\mathcal{C}(Y)} H^k_{(2)}(X;LG)$$

and the L^2-Betti numbers of Y with respect to G to be

$$\beta^k_{(2)}(Y;LG) = \dim_{LG} H^k_{(2)}(Y;LG).$$

In particular, the properties of the dimension function with respect to the colimit shows that if $Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y$ is an exhaustion of Y by finite G-CW-subcomplexes, then

$$\beta^k_{(2)}(Y; LG) = \sup_i \inf_{j>i} \dim_{LG} \operatorname{im}(H^n_{(2)}(Y_i) \to H^n_{(2)}(Y_j)).$$

If Y is already finite, then this clearly coincides with the previous definition.

To see that the definitions of Lück and Cheeger and Gromov coincide, note first that if $f : X \to Y$ then

$$\dim_{LG} \operatorname{im}(f_* : H^{(2)}_n(X) \to H^{(2)}_n(Y)) = \dim_{LG} \operatorname{im}(f^* : H^n_{(2)}(Y) \to H^n_{(2)}(X)).$$

Indeed, through the isomorphisms given above, these two maps are each other's adjoint, and hence

$$\begin{aligned}\dim_{LG} \overline{\operatorname{im} f_*} &= \dim_{LG} (\ker f^*)^\perp = \dim_{LG} H^k_{(2)}(Y) - \dim_{LG} \ker f^* \\ &= \dim_{LG} \overline{\operatorname{im} f^*}.\end{aligned}$$

Thus

$$\begin{aligned}\beta^k_{(2)}(Y; LG) &= \sup_i \inf_{j>i} \dim_{LG} \operatorname{im}(H^{(2)}_n(Y_i; LG) \to H^{(2)}_n(Y_j; LG)) \\ &= \dim_{LG} H_n(LG \otimes_{\mathbb{C}G} C^{\text{sing}}_*(Y)) \\ &= \beta^{(2)}_k(Y; LG)\end{aligned}$$

by Lemmas 2.3 and 2.4, and the behavior of the dimension under colimits.

2.2 Properties of L^2-Betti numbers of groups

We will summarize some of the properties of L^2-Betti numbers of groups.

Lemma 2.7. *For a group G, we have*

$$\beta^{(2)}_0(G) = |G|^{-1}$$

where $|G|^{-1} = 0$ if G is infinite.

Lemma 2.8. *Let G_1 and G_2 be nontrivial groups, and let $G = G_1 * G_2$. Then*

$$\beta^{(2)}_n(G) = \begin{cases} 0 & \text{if } n = 0 \\ \beta^{(2)}_1(G_1) - \beta^{(2)}_0(G_1) + \beta^{(2)}_1(G_2) - \beta^{(2)}_0(G_2) + 1 & \text{if } n = 1 \\ \beta^{(2)}_n(G_1) + \beta^{(2)}_n(G_2) & \text{if } n \geq 2. \end{cases}$$

Proof. Recall that $BG = BG_1 \vee BG_2$, so that we get an exact sequence of chain complexes

$$\begin{aligned} 0 \longrightarrow\ & LG \otimes_{\mathbb{C}} C_*(\{*\}) \longrightarrow \\ & \longrightarrow LG \otimes_{\mathbb{Z}G_1} C_*(EG_1) \oplus LG \otimes_{\mathbb{Z}G_2} C_*(EG_2) \longrightarrow \\ & \qquad \longrightarrow LG \otimes_{\mathbb{Z}G} C_*(EG) \longrightarrow 0 \end{aligned}$$

and a corresponding long exact sequence in homology. Since $C_n(\{*\})$ is concentrated in dimension zero, this sequence breaks down into

$$0 \longrightarrow \left(F_1 H_n^{(2)}(G_1)\right) \oplus \left(F_2 H_n^{(2)}(G_2)\right) \longrightarrow H_n(G) \longrightarrow 0$$

for $n \geq 2$, and for the low-dimensional terms,

$$0 \longrightarrow \left(F_1 H_1^{(2)}(G_1)\right) \oplus \left(F_2 H_1^{(2)}(G_2)\right) \longrightarrow H_1^{(2)}(G) \longrightarrow$$

$$\longrightarrow LG \longrightarrow \left(F_1 H_0^{(2)}(G_1)\right) \oplus \left(F_2 H_0^{(2)}(G_2)\right) \longrightarrow 0$$

since G is infinite, where $F_1 = LG \otimes_{LG_1}$ and $F_2 = LG \otimes_{LG_2}$. Hence, the result follows from additivity of the dimension function and its properties under induction. $\square$

Recall that two groups G_1 and G_2 are said to be *orbit equivalent* if there is a measure space (X, μ) and free measure-preserving actions of G_1 and G_2 on (X, μ) which induce the same equivalence relation.

Theorem 2.9 (Gaboriau). *Let G_1 and G_2 be orbit equivalent groups. Then*

$$\beta_n^{(2)}(G_1) = \beta_n^{(2)}(G_2)$$

for all $n \geq 0$.

The proof can be found in [11, Theorem 3.2], with an alternative proof given in [17] and [22].

Corollary 2.10. *If G is infinite amenable, then*

$$\beta_n^{(2)}(G) = 0$$

for all $n \geq 0$.

Proof. By [6] any infinite amenable group is orbit equivalent to the group $\mathbb{Z}$ of integers. Furthermore, we have already seen that $\beta_n^{(2)}(\mathbb{Z}) = 0$ for all $n \geq 0$. $\square$

A more direct proof can be found in [4, Theorem 0.2] or (somewhat stronger) in [13, Theorem 5.1].

Remark. Lück in [14, p. 41] constructs groups $G_i(m,n)$ for integers $i,n \geq 1$ and $m \geq 1$ such that

$$\beta_k^{(2)}(G_i(m,n)) = \begin{cases} \frac{m}{n} & \text{if } k = i \\ 0 & \text{otherwise.} \end{cases}$$

It is an open question whether $\beta_k^{(2)}(G)$ is always rational.

Chapter 3

L^2-Betti numbers of Equivalence Relations

In this section we will present the definitions of L^2-Betti numbers of countable standard equivalence relations R given by Gaboriau [11] and Sauer [17], and show that they coincide. First, however, we must recall some concepts related to equivalence relations, following [8, 9, 11, 17].

3.1 Basic notions

Let (X, μ) be a standard measure space, and let $R \subset X \times X$ be an equivalence relation on X. We say that R is *countable standard* if the set

$$\{y : (x, y) \in R\}$$

is countable for each $x \in X$, and if furthermore R is a Borel subset of $X \times X$. Say that R *preserves* μ if every partial Borel isomorphism $\phi : A \to B$ for $A, B \subset X$ with graph contained in R preserves μ.

Feldman and Moore [8, Theorem 1] have shown that for any countable standard μ-preserving equivalence relation R, there is a μ-preserving action of a discrete countable group G on X inducing R. However, one cannot always choose the action to be free, as shown by Furman [10].

Let us now recall the construction of [9] of the von Neumann algebra LR of the relation R. First, say that a function $f \in L^\infty(R)$ is *left bounded* if

$$x \mapsto \#\{y : f(x, y) \neq 0\} + \#\{y : f(y, x) \neq 0\}$$

is in $L^\infty(X)$. Give the set of left bounded functions pointwise addition, and a multiplication given by

$$(f * g)(x, z) = \sum_{y \sim x} f(x, y) g(y, z).$$

We write $\mathbb{C}R$ for the resulting ring, since it is the groupoid ring of the measurable groupoid R. This ring has a natural action of left multiplication on $L^2(R)$, and we let LR be the von Neumann algebra generated by this representation. (Note that the characteristic function χ_D of the diagonal $D \subset R$ is a cyclic and separating vector for LR, so that $L^2(R) = L^2(LR)$.) In the case where R is induced by the free action of a group G, we have $LR = L^\infty(X) \rtimes G$.

As an aside, we note the following.

Lemma 3.1. *Let A be a $\mathbb{C}R$-module. Then $\mathbb{C}R$ acts on A as Lipschitz maps with respect to the rank metric on A induced by $L^\infty(X) \subset \mathbb{C}R$.*

Proof. Let $f \in \mathbb{C}R$, and let N be such that

$$\#\{x : f(x,y) \neq 0\} \leq N$$

for all $y \in X$. Then if $\xi \in A$ and $\chi_E \xi = \xi$, let $F \subset X$ be given by

$$F = \{x \in X : \exists y \in E, f(x,y) \neq 0\}.$$

Then $\mu(F) \leq N\mu(E)$ and $\chi_F f\xi = f\xi$, so $[f\xi] \leq N[\xi]$. $\square$

3.2 Simplicial R-complexes

A *standard fiber space* on X is a Borel space U together with a Borel map $\pi : U \to X$ with countable fibers. There is then a natural measure ν_U on (U, π) given by

$$\nu_U(C) = \int_X \#(\pi^{-1}(x) \cap C) d\mu(x).$$

The example that we will be the most concerned with in the following is that where $U = R$, and π is either π_l or π_r, the projections onto the first and second coordinates respectively. By the invariance of R, the the induced measures on R are the same, denoted simply ν.

Now, given two standard fiber spaces on X, (U, π) and (V, π'), we may form their fiber product

$$U * V = \{(u,v) \in U \times V : \pi(u) = \pi'(v)\}$$

which is again a standard fiber space.

Next, a standard fiber space U on X may have a left R-action. That is, a Borel map $(R, \pi_r) * U \to U$ denoted $((x,y),u) \mapsto (x,y)u$ where $y = \pi(u)$, satisfying

$$(x,y)((y,z)u) = (x,z)u, \qquad (z,z)u = u$$

whenever this makes sense. This implies that $\pi((y,z)u) = y$, and that (x,y) is a bijection between $\pi^{-1}(y)$ and $\pi^{-1}(x)$.

For $u \in U$, write Ru for the *orbit* of u, that is,

$$Ru = \{(x, \pi(u))u : x \sim \pi(u)\}.$$

Say that the R-action on U is *discrete* if there is a Borel fundamental domain for the action. That is, if there is a Borel set $F \subset U$ intersecting each orbit once and only once. For the case $U = R$, the diagonal D is a fundamental domain for the standard R-action.

Note that if F is a fundamental domain for U then $F * V$ is a fundamental domain for $U * V$. Also, if F is a fundamental domain, we may write

$$F = \bigsqcup_{j \in J} F_j$$

where J is countable, in such a manner that the F_j are Borel and $\pi|_{F_j}$ is injective.

Lemma 3.2. *Any discrete standard fiber space U can be embedded into $\bigsqcup_{i=1}^{\infty} R$ in a way compatible with the R-structure.*

Proof. Let F be a fundamental domain for U. If we split F into F_j as above, where $\pi : U \to X$ is injective when restricted to F_j, we can identify RF_j with RD_j where $D_j = \pi(F_j) \subset R$ is part of the diagonal of R. Since there are only countably many such F_j, we are done. $\square$

Definition 3.3. A simplicial R-complex Σ consists of a discrete R-space Σ^0 and Borel sets $\Sigma^1, \Sigma^2, \ldots$ with

$$\Sigma^n \subset \overbrace{\Sigma^0 * \cdots * \Sigma^0}^{n+1}$$

satisfying, for $n > 0$,

(i) $R\Sigma^n = \Sigma^n$;
(ii) $(v_0, \ldots, v_{n+1}) \in \Sigma^{n+1}$ implies $(v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}) \in \Sigma^n$;
(iii) Σ^n is invariant under permutation of coordinates;
(iv) if $(v_0, \ldots, v_n) \in \Sigma^n$ then $v_i \neq v_j$ for $i \neq j$.

These conditions are there mainly to assure that Σ^n is an R-fiber space, and that there is a well defined boundary operator on Σ.

Example 3.4. Let R be an equivalence relation. We wish to construct what is, in a sense, the largest possible simplicial R-complex. To do this, let

$$ER^0 = \{(x, y, i) : x \sim y, i \in \mathbb{N}\} = \bigsqcup_{i=1}^{\infty} R,$$

and

$$ER^n \subset \overbrace{ER^0 * \cdots * ER^0}^{n+1}.$$

consist of the elements of the form $(x, (y_0, i_0), \ldots, (y_n, i_n))$ where $x \sim y_k$, $i_k \in \mathbb{N}$ and $(y_k, i_k) \neq (y_l, i_l)$.

To see that this is the "largest" R-complex, consider a simplicial R-complex Σ. We may then embed Σ^0 into ER^0 by the lemma above and so get $\Sigma^n \subset ER^n$ for all n.

A fundamental domain for ER^0 is given by

$$\{(x, (x, i)) : i \in \mathbb{N}\}$$

and can be given for ER^n by first choosing a total order on $R \times \mathbb{N}$ and then setting

$$\{(x, (y_0, i_0), \ldots, (y_n, i_n)) : y_0 = x, (y_k, i_k) < (y_{k+1}, i_{k+1})\}.$$

Say that Σ is ULB or *uniformly locally bounded* if there is an integer N such that (ν-almost) every $h \in \Sigma^0$ is a vertex in no more than N simplices, and if furthermore Σ^0 has a fundamental domain of finite measure.

Also, given a simplicial R-complex Σ, we may associate to it a field of simplicial complexes by for $x \in X$ letting Σ_x^k be the fiber of $\pi : \Sigma^k \to X$ over x. We say that Σ is *n-dimensional*, *contractible*, and so on, if these properties hold for Σ_x for (ν-almost) every $x \in X$.

Note in particular that any ULB Σ is finite-dimensional.

Lemma 3.5. *If Σ is ULB, then for all n there is a fundamental domain for Σ^n of finite measure.*

Proof. Embed Σ into ER, and note that the fundamental domain for Σ^0 given by

$$F^0 = \{(x, (x, i)) : i \in \mathbb{N}\} \cap \Sigma^0$$

has finite measure by definition. Now, if N is the integer from the definition of a ULB complex, then the fundamental domain

$$F^n = \{(x, (y_0, i_0), \ldots, (y_n, i_n)) : y_0 = x, (y_k, i_k) < (y_{k+1}, i_{k+1})\} \cap \Sigma^n$$

satisfies $\nu(F^n) \leq N \cdot \nu(F^0)$, since F^n contains less than N simplices with $(x, (x, i))$ as a corner for all x and i. □

3.3 Chains of R-complexes

Let us now start stepping towards our definitions of homology of equivalence relations by first defining what we mean by the n-chains of a simplicial R-complex Σ.

First, note that every Borel section s of the fibration $\pi : \Sigma^n \to X$ defines a vector field $x \mapsto s(x)$ with values in $C_n(\Sigma_x, \mathbb{Z}) \subset C_n^{(2)}(\Sigma_x)$. Say that a field of vectors $x \mapsto \sigma(x) \in C_n^{(2)}(\Sigma)$ is *Borel* if $x \mapsto \langle \sigma(x), s(x) \rangle_x$ is Borel for every Borel section s.

Definition 3.6. Define the following spaces of n-chains:

(i) $C_n^{(2)}(\Sigma)$, the space of L^2-chains, consists of all Borel vector fields $x \mapsto \sigma(x)$ such that $\sigma(x) \in C_n^{(2)}(\Sigma_x)$ and such that $x \mapsto \|\sigma(x)\|$ is an element of $L^2(X)$;

(ii) $C_n(\Sigma)$, the space of chains, consists of all Borel vector fields $x \mapsto \sigma(x)$ such that $\sigma(x) \in C_n(\Sigma_x)$.

In both cases we must choose an orientation on the n-simplices, which can be done by specifying a fundamental domain for the action of S_{n+1} on Σ^n. Note that we need to assume that Σ is ULB to have well defined boundary operators in the first case, similarily to the case for CW-complexes.

There is a set of n-chains that is given as soon as a fundamental domain F for the R-action on Σ^n is chosen. Indeed, let $F = \bigsqcup_{j=1}^{\infty} F_j$ be a Borel splitting of F such that $\pi : \Sigma^n \to X$ is injective on each F_j, and define $s_j \in C_n(\Sigma)$ by setting

$$s_j(x) = \begin{cases} 0 & \text{if } x \notin \pi(F_j); \\ \pi^{-1}(x) \cap F_j & \text{otherwise.} \end{cases}$$

Then the images $(x, y)s_j(y)$ for $y \sim x$ exhaust Σ_x^n, and the correspondence is injective. Let us for the rest of this chapter put $p_i = \chi_{\pi(F_i)}$, so that s_i is supported on the same set as p_i. Note that $p_i \in L^\infty(X) \subset LR$.

Proposition 3.7. *The inclusion $\bigcup_{n\geq 1} \bigoplus_{i=1}^{n} \mathbb{C}Rp_i s_i \subset C_n(\Sigma)$ is an isomorphism after completion with respect to the rank metric induced by $L^\infty(X)$.*

Proof. Since $\Sigma_x^n = \{(x, y)s_i(y) : y \sim x, i \in \mathbb{N}\}$, any section $\sigma \in C_k(\Sigma)$ can be written as

$$\sigma(x) = \sum_{i=1}^{\infty} \sum_{y \sim x} f_i(x, y)(x, y)s_i(y)$$

for some $f_i \in M(R)$, where the sum only includes finitely many nonzero terms for each x. (Note that these are not sufficient conditions for σ to be in $C_k(\Sigma)$.)

Write P for the $\mathbb{C}R$-module $\bigcup_{n\geq 1} \bigoplus_{i=1}^{n} \mathbb{C}Rp_i s_i$. We will first show that P is rank dense in the set of $\sigma \in C_k(\Sigma)$ which can be written in the form

$$\sigma(x) = \sum_{i=1}^{n} \sum_{y \sim x} f_i(x, y)(x, y)s_i(y),$$

and then show that the set of such σ is rank dense in $C_k(\Sigma)$.

For the first part, write R as the disjoint union of a countable set of local Borel isomorphisms ϕ_i. This can be done for instance by assuming that $R = R_G$ for the action of some discrete countable group $G = \{g_1, g_2, \ldots\}$

and setting $\phi_i(x) = g_i x$ on their domains, where ϕ_i has domain the set of $x \in X$ such that $g_i x \neq g_j x$ for $j < i$. Then for any $m \geq 1$ there is a set X_m such that

$$\sigma(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(x, \phi_j(x))(x, \phi_j(x)) s_i(\phi_j(x))$$

for $x \in X_m$. On this set σ coincides with an element of P, since the f_i clearly are elements of $\mathbb{C}G$ here. Next, $\lim_m \mu(X_m) = \mu(X)$, so for each $\epsilon > 0$, there is some m with $\mu(X_m) \geq \mu(X) - \epsilon$. Hence s is in the rank closure of P.

Now, if $\sigma \in C_k(\Sigma)$ is given by

$$\sigma(x) = \sum_{i=1}^{\infty} \sum_{y \sim x} f_i(x, y) s_i(y)$$

as above, consider the sets

$$X_n = \{x \in X : \sigma(x) = \sum_{i=1}^{n} \sum_{y \sim x} f_i(x, y) s_i(y)\}.$$

Then $\sigma|_{X_n}$ is in the rank closure of P with respect to $L^\infty(X)$ by the last paragraph, and since $\lim_{n\to\infty} \mu(X_n) = 1$, we have that σ is in the rank closure of P. Hence P is $L^\infty(X)$-dense in $C_n(\Sigma)$. □

Proposition 3.8. *If Σ is ULB, then there is an isomorphism between the completions of $LR \otimes_{\mathbb{C}R} C_n(\Sigma)$ and $C_n^{(2)}(\Sigma)$ which respects the boundary operators.*

Proof. In the last lemma, we found an $L^\infty(X)$-dense submodule

$$P = \bigcup_{n \geq 1} \bigoplus_{i=1}^{n} \mathbb{C}R p_i s_i$$

of $C_*(\Sigma)$, whence $LR \otimes_{\mathbb{C}R} P$ is LR-dense in $LR \otimes_{\mathbb{C}R} C_*(\Sigma)$. We want to show that $LR \otimes_{\mathbb{C}R} P$ is LR-dense in $C_n^{(2)}(\Sigma)$.

First note that $LR \otimes_{\mathbb{C}R} P$ is LR-dense in

$$L^2(R) \otimes_{\mathbb{C}R} P = \bigcup_{n \geq 1} \bigoplus_{i=1}^{n} L^2(R) p_i s_i.$$

On the other hand, elements of $C_n^{(2)}(\Sigma)$ are given by sums

$$\sigma(x) = \sum_{i=1}^{\infty} \sum_{y \sim x} f_i(x, y)(x, y) s_i(y)$$

where the f_i satisfy $\sum_{i=1}^{\infty} \|f_i\|_2^2 < \infty$. Hence we have an isomorphism

$$C_n^{(2)}(\Sigma) \simeq \bigoplus_{i=1}^{\infty} L^2(R) p_i s_i.$$

Now, since Σ is ULB, Σ^k has a fundamental domain F of finite measure, and since

$$\nu(F) = \sum_{i=1}^{\infty} \mu(\pi(F_i)) = \sum_{i=1}^{\infty} \tau(p_i)$$

there is an integer N such that $\sum_{i=N}^{\infty} \tau(p_i) < \epsilon$. Hence

$$[\sigma - \sum_{i=1}^{N-1} f_i p_i s_i] = [\sum_{i=N}^{\infty} f_i p_i s_i] \leq \sum_{i=N}^{\infty} \tau(p_i) < \epsilon,$$

so that $L^2(R) \otimes_{\mathbb{C}R} P$ is rank dense in $C_n^{(2)}(\Sigma)$. Hence the completions are isomorphic as claimed. $\square$

Corollary 3.9. *If Σ is a ULB R-complex, then*

$$\dim_{LR} C_n^{(2)}(\Sigma) < \infty$$

for all n.

Proof. In the above proof, we noted that

$$C_n^{(2)}(\Sigma) \simeq \bigoplus_{i=1}^{\infty} L^2(R) p_i s_i$$

where the p_i are projections such that $\sum_{i=1}^{\infty} \tau(p_i) < \infty$. Hence

$$\dim_{LR} C_n^{(2)}(\Sigma) = \sum \tau(p_i) < \infty.$$

$\square$

3.4 L^2-homology of equivalence relations

Let us recall the definitions of L^2-Betti numbers for equivalence relations given in [11] and [17].

Definition 3.10 (Sauer). Given an equivalence relation R, define the n-th L^2-Betti number of R to be

$$\beta_n^{(2)}(R) = \dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, L^\infty(X)).$$

Definition 3.11 (Gaboriau). Let R be an equivalence relation, let Σ be a contractible simplicial R-complex, and let Σ_α be an exhaustion of Σ by ULB subcomplexes. Then define the n-the L^2-Betti number of R to be

$$\beta_n^{(2)}(R) = \sup_\alpha \inf_{\beta \geq \alpha} \dim_{LR} \operatorname{im}(H_n^{(2)}(\Sigma_\alpha, R) \to H_n^{(2)}(\Sigma_\beta, R))$$

where $H_n^{(2)}(\Sigma_\alpha, R)$ is the reduced L^2-homology of the chain complex $C_n^{(2)}(\Sigma)$.

We wish to show that these two definitions coincide, and take that of Sauer as our starting point.

Since $\mathbb{C}R$ acts on $L^\infty(X)$ as Lipschitz functions, we may replace $L^\infty(X)$ by its rank completion $M(X)$, to leave the problem of calculating

$$\dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, M(X)).$$

Let us first do some auxiliary work.

Let X be a standard measure space, and let V be a vector space over $\mathbb{Q}$ of countable dimension.

Definition 3.12. Say that a field $x \mapsto V_x$ of subspaces of V is *measurable* if for all measurable mappings $s : X \to V$ the set

$$\{x \in X : s(x) \in V_x\}$$

is measurable. Likewise, say that a field $x \mapsto T_x$ of operators on V is *measurable* if $x \mapsto T_x s(x)$ is measurable for every measurable mapping $s : X \to V$.

Lemma 3.13. *Let $x \mapsto V_x$ be a measurable field of subspaces of V. Then there is a measurable field of projections $x \mapsto p_x$ such that $p_x : V \to V$ is a projection onto V_x.*

Proof. Let $\{e_1, e_2, \ldots\}$ be a basis for V, and set $V_n = \operatorname{Span}_{\mathbb{Q}}\{e_1, \ldots, e_n\}$. We want to define p_x inductively on each V_n.

On V_1 this is trivial, as $p_x e_1 = \chi_{V_x}(e_1) e_1$ is the only possible option for p_x.

Hence, assume by induction that $p_x : V_{n-1} \to V_{n-1}$ is defined and measurable, and is a projection onto $V_{n-1} \cap V_x$. Let $X' \subset X$ be the set of those x such that $V_{n-1} \cap V_x$ is strictly included in $V_n \cap V_x$. Let $U_1, U_2, \ldots$ be an enumeration of the subspaces of V_n, and let $X_i \subset X'$ be the set of x such that $V_n \cap V_x = U_i$. This set is measurable. Now for $v \in V_{n-1}$, let $X_v \subset X_i$ be the set of $x \in X$ such that $e_n + v \in V_x$ and $p_x v = 0$. Then, after choosing an ordering on V_{n-1}, we may put

$$p_x e_n = \begin{cases} e_n + v_1 & \text{for } x \in X_{v_1} \\ e_n + v_j & \text{for } x \in X_j \setminus \bigcup_{k<j} X_{v_k}. \end{cases}$$

Doing this for all i, we have defined $p_x : V_n \to V_n$ in a measurable way such that p_x is a projection onto $V_n \cap V_x$, as long as we put $p_x e_n = 0$ on $X \setminus X'$.

By induction, the field of operators $p_x : V \to V$ is then measurable, and is a projection onto V_x for all $x \in X$. □

Lemma 3.14. *Let $T_x : V \to V$ be a measurable field of operators, let V_x and W_x be measurable fields of subspaces of V, and assume that $T_x W_x = V_x$ for all x. Then there is a measurable field of operators $h_x : V \to V$ such that h_x maps V into W_x, and which satisfies*

$$T_x h_x = p_x \qquad h_x T_x = (1 - q_x)$$

where p_x and q_x are measurable fields of projections onto V_x and $\ker T_x \cap W_x$ respectively.

Proof. Let as before $\{e_1, e_2, \ldots\}$ be a basis for V, and enumerate V as $V = \{v_1, v_2, \ldots\}$. For $i, j \in \mathbb{N}$, put

$$X_{ij} = \{x \in X : v_i \in (1 - q_x)W_x, T_x v_i = p_x e_j\}.$$

Then the X_{ij} are measurable with $\bigsqcup_{i=1}^{\infty} X_{ij} = X$ for all $j \in \mathbb{N}$, and so the field of operators given by

$$h_x e_j = v_i \qquad \text{for } x \in X_{ij}$$

is measurable. Furthermore, it clearly has the stated properties. □

From this we get the following.

Theorem 3.15. *Let Σ be an n-connected simplicial R-complex. Then the sequence*

$$0 \longleftarrow M(X) \longleftarrow C_0(\Sigma) \longleftarrow \cdots \longleftarrow C_n(\Sigma) \longleftarrow C_{n+1}(\Sigma)$$

is exact.

Proof. Let us first show this for the same sequence with rational coefficients. Hence consider the piece

$$C_{k-1}(\Sigma, \mathbb{Q}) \xleftarrow{\partial_k} C_k(\Sigma, \mathbb{Q}) \xleftarrow{\partial_{k+1}} C_{k+1}(\Sigma, \mathbb{Q})$$

of the sequence. Embedding Σ into ER, we may identify the $C_*(\Sigma, \mathbb{Q})$ with measurable fields $x \mapsto C_*(\Sigma_x, \mathbb{Q})$ of subspaces of the $\mathbb{Q}$-vector space V of countably infinite dimension.

If we now put $V_x = \ker \partial_{k,x}$, $W_x = C_{k+1}(\Sigma_x, \mathbb{Q})$ and $T_x = \partial_{k+1,x}$, we are in the situation of the above lemma. Hence we can construct a homotopy $h_k : C_k(\Sigma, \mathbb{Q}) \to C_{k+1}(\Sigma, \mathbb{Q})$ such that for $k = 1, \ldots, n$

$$\mathrm{id} = h_{k-1}\partial_k + \partial_{k+1}h_k.$$

Hence the sequence is exact.

To move to complex coefficients, we now only have to note that any element of $C_k(\Sigma)$ is nothing but an element of $C_k(\Sigma, \mathbb{Q})$ multiplied with a function in $L^\infty(X)$, and since ∂_k and the lift of h_k are $L^\infty(X)$-module homomorphisms, the sequence is still exact. $\square$

Lemma 3.16. *For an R-complex Σ, we have*

$$\dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, C_k(\Sigma)) = 0$$

for $n \geq 1$.

Proof. Recall that there is a $\dim_{L^\infty(X)}$-isomorphism

$$P = \bigcup_{n \geq 1} \bigoplus_{i=1}^n \mathbb{C}Rp_i s_i \longrightarrow C_k(\Sigma).$$

Since $\mathbb{C}R$ acts by Lipschitz operators this implies that

$$\dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, C_k(\Sigma)) = \dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, P)$$

which is zero for $n \geq 1$ as P is projective. $\square$

From this and Lemma 1.31 we get that

$$\dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, M(X)) = \dim_{LR} H_n(LR \otimes_{\mathbb{C}R} C_*(ER)).$$

Since we by Proposition 3.8 have

$$\dim_{LR} H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma)) = \dim_{LR} H_n(C_*^{(2)}(\Sigma))$$

if Σ is ULB, it remains to show that

$$\dim_{LR} H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma)) = \dim_{LR} \lim_{\alpha} H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma_\alpha))$$

for an exhaustion Σ_α of Σ by ULB subcomplexes.

For this, recall that there is an $L^\infty(X)$-dense submodule $P \subset C_n(\Sigma)$ with elements of the form

$$\sigma(x) = \sum_{i=1}^n \sum_{y \sim x} f_i(x, y)(x, y) s_i(y)$$

with $f_i \in \mathbb{C}R$. Since we only involve finitely many s_i, we have $s \in C_n(\Sigma_\alpha)$ for some ULB $\Sigma_\alpha \subset \Sigma$. Hence the union $\bigcup_\alpha C_n(\Sigma_\alpha)$ is $L^\infty(X)$-dense in $C_n(\Sigma)$.

By lemma 1.27 this implies that the inclusion

$$LR \otimes_{\mathbb{C}R} \bigcup_\alpha C_n(\Sigma_\alpha) \longrightarrow LR \otimes_{\mathbb{C}R} C_n(\Sigma)$$

is a $\dim_{LR}$-isomorphism for each n. Hence we have $\dim_{LR}$-isomorphisms

$$H_n(LR \otimes_{\mathbb{C}R} \bigcup_{\alpha} C_n(\Sigma_\alpha)) \longrightarrow H_n(LR \otimes_{\mathbb{C}R} C_n(\Sigma)),$$

and recalling that $H_n(LR \otimes_{\mathbb{C}R} \bigcup_\alpha C_n(\Sigma_\alpha)) = \lim_\alpha H_n(LR \otimes_{\mathbb{C}R} C_n(\Sigma_\alpha))$ we get

$$\dim_{LR} H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma)) = \dim_{LR} \operatorname{colim}_\alpha H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma_\alpha)).$$

Now recall that since Σ_α is ULB, we have $\dim_{LR} C_n^{(2)}(\Sigma) < \infty$, and hence

$$\dim_{LR} H_n^{(2)}(C_n^{(2)}(\Sigma_\alpha)) = \dim_{LR} H_n(C_n^{(2)}(\Sigma_\alpha))$$

by Lemma 2.4. We can then tie up our chain of equalities to see that

$$\begin{aligned}
\dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, L^\infty(X)) &= \dim_{LR} \operatorname{Tor}_n^{\mathbb{C}R}(LR, M(X)) \\
&= \dim_{LR} H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma)) \\
&= \dim_{LR} \operatorname{colim}_\alpha H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma)) \\
&= \sup_\alpha \inf_{\beta \geq \alpha} \dim_{LR} H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma)) \\
&= \sup_\alpha \inf_{\beta \geq \alpha} \dim_{LR} H_n(C_*^{(2)}(\Sigma)) \\
&= \sup_\alpha \inf_{\beta \geq \alpha} \dim_{LR} H_n^{(2)}(C_*^{(2)}(\Sigma))
\end{aligned}$$

which is exactly Gaboriau's definition of the n-th L^2-Betti number of R. Note that we used that since $\dim_{LR} LR \otimes_{\mathbb{C}R} C_*(\Sigma_\alpha) < \infty$ for Σ_α ULB we have $\dim_{LR} H_n(LR \otimes_{\mathbb{C}R} C_*(\Sigma_\alpha)) < \infty$, so that we can expand the dimension of the colimit.

Hence we have proved the following result.

Theorem 3.17. *The definitions of L^2-Betti numbers for a standard countable equivalence relation given by Gaboriau and Sauer agree.*

Our argument also gives an alternative proof of the first part of [11, Theorem 3.13].

Theorem 3.18 (Gaboriau). *If Σ is an n-connected simplicial R-complex, then*

$$\beta_k^{(2)}(\Sigma, R) = \beta_k^{(2)}(R)$$

for $0 \leq k \leq n$.

Proof. Let Σ be n-connected. Then we may embed Σ into a contractible R-complex $\tilde{\Sigma}$ such that $\Sigma^k = \tilde{\Sigma}^k$ for $k \leq n+1$, whence we have $\beta_k^{(2)}(\Sigma, R) = \beta_k^{(2)}(\tilde{\Sigma}, R)$ for $k \leq n$. But $\beta_k^{(2)}(\tilde{\Sigma}, R) = \beta_k^{(2)}(R)$, so we are done. ☐

Chapter 4

L^2-Betti numbers of von Neumann Algebras

Given a group G, we have shown that

$$\beta_n^{(2)}(G) = \dim_{LG} \operatorname{Tor}_n^{\mathbb{C}G}(LG, \mathbb{C}).$$

This involves two representations of G, the trivial representation, denoted $1_G : G \to U(\mathbb{C})$, and the left regular representation $\lambda_G : G \to U(\ell^2(G))$. The main idea of Connes and Shlyakhtenko's definition [7] of L^2-Betti numbers of tracial algebras is to combine this observation with the links between representations of groups and bimodules over von Neumann algebras expressed in Connes' theory of correspondences, as expressed for instance in [5, Appendix 5B].

What we need from that theory is the following "dictionary", which we will defend immediately.

Group	Finite von Neumann algebra
Unitary representation	M-bimodule
Trivial representation	$L^2(M)$
Left regular representation	$L^2(M)\bar{\otimes}L^2(M)$

The first link consists of the usual construction of a group von Neumann algebra LG from G. This is finite, since it inherits its trace from the representation on $\ell^2(G)$.

For the second link, let $\pi : G \to U(H)$ be a unitary representation of G. Then, by replacing H with $H\bar{\otimes}\ell^2(G)$, we may create left and right representations of G given by

$$\pi \otimes \lambda, \qquad 1 \otimes \rho$$

where λ and ρ are the left and right regular representations, respectively. These representations of G make $H\bar{\otimes}\ell^2(G)$ into a $\mathbb{C}G$-bimodule, and the structure extends by continuity to a structure of LG-bimodule.

Establishing the third and fourth lines of the "dictionary" now consists in carrying out this construction for the given representations. Let us first consider the trivial representation $1_G : G \to U(\mathbb{C})$. Then $H\bar{\otimes}\ell^2(G) = \ell^2(G)$, and the representations are exactly the left and right regular representations. Since $L^2(LG) = \ell^2(G)$ and the induced module structure is exactly the standard action, this explains the "translation".

Finally, if we consider the left regular representation, we get to consider the representations

$$\lambda \otimes \lambda, \qquad 1 \otimes \rho$$

of G on $\ell^2(G)\bar{\otimes}\ell^2(G)$. We claim that these representations are equivalent to the representations $\lambda \otimes 1$ and $1 \otimes \rho$ on the same Hilbert space.

Indeed, define $\phi : \ell^2(G) \otimes \ell^2(G) \to \ell^2(G) \otimes \ell^2(G)$ by

$$\phi(\delta_g \otimes \delta_h) = \delta_h \otimes \delta_{g^{-1}h}.$$

Then we clearly have $\phi \circ (\lambda \otimes \lambda) = (\lambda \otimes 1) \circ \phi$ and $\phi \circ (1 \otimes \rho) = (1 \otimes \rho) \circ \phi$.

Lemma 4.1. *Let M and N be von Neumann algebras, and let A be an $M \otimes N^o$-module. Then $M \otimes N^o$ acts on A as Lipschitz operators with respect to the rank metric induced by the left action of M.*

Proof. Let $m \in M$, $n \in N$ and $a \in A$. Then $[(1 \otimes n^o)a]_M \leq [a]_M$ by Lemma 1.12(i) and $[(m \otimes 1^o)a]_M \leq [a]_M$ by Lemma 1.13. Hence, for an element $x = \sum_{i=1}^k m_i \otimes n_i^o \in M \otimes N^o$ we have

$$[xa]_M \leq \sum_{i=1}^k [(m_i \otimes n_i^o)a]_M \leq k[a]_M,$$

whence the action is Lipschitz. □

Hence we may "translate" the definition of Lück, to get

$$\begin{aligned}
\beta_n^{(2)}(LG) &= \dim_{LG\bar{\otimes}LG^o} \operatorname{Tor}_n^{LG\otimes LG^o}(\ell^2(G)\bar{\otimes}\ell^2(G), \ell^2(G)) \\
&= \dim_{LG\bar{\otimes}LG^o} \operatorname{Tor}_n^{LG\otimes LG^o}(L^2(LG\bar{\otimes}LG^o), \ell^2(G)) \\
&= \dim_{LG\bar{\otimes}LG^o} \operatorname{Tor}_n^{LG\otimes LG^o}(LG\bar{\otimes}LG^o, \ell^2(G)) \\
&= \dim_{LG\bar{\otimes}LG^o} \operatorname{Tor}_n^{LG\otimes LG^o}(LG\bar{\otimes}LG^o, LG)
\end{aligned}$$

by Propositions 1.25 and 1.30. After replacing LG by a general finite von Neumann algebra M, we hence get as in [7] a definition

$$\beta_n^{(2)}(M) = \dim_{M\bar{\otimes}M^o} \operatorname{Tor}_n^{M\otimes M^o}(M\bar{\otimes}M^o, M)$$

of L^2-Betti numbers of von Neumann algebras. More generally, we have the following.

Definition 4.2. Let (A, τ) be a tracial $*$-algebra, and let M be the von Neumann algebra generated by the GNS-representation of A associated to the trace. Then define

$$\beta_n^{(2)}(A, \tau) = \dim_{M \bar{\otimes} M^o} \operatorname{Tor}_n^{A \otimes A^o}(M \bar{\otimes} M^o, A).$$

Remark. Some basic properties of L^2-Betti numbers of von Neumann algebras are demonstrated in [7, 20, 21, 22]. However, beyond $\beta_0^{(2)}(M)$ and the L^2-Betti numbers of finite-dimensional and commutative von Neumann algebras (which are trivial for $n \geq 1$), no calculations have been published.

4.1 Relativizing the definition

It would be interesting to have a "relative" notion of L^2-Betti numbers of von Neumann algebras. Since it has been asserted [7, 15] that the definition

$$\beta_n^{(2)}(G) = \dim_{LG} \operatorname{Tor}_n^{L^\infty(X) \rtimes_{\text{alg}} G}(L^\infty(X) \rtimes G, L^\infty(X))$$

of Sauer [17] and Thom [22] is in a sense "relative", it might prove a suitable starting point.

Hence, let us take this as our starting point, and then to emulate the constructions of Connes and Shlyakhtenko. Hence, we look at representations of the algebraic crossed product $N \rtimes_{\text{alg}} G$, and want to move to bimodules over appropriate von Neumann algebras. Write $M = N \rtimes G$ for brevity.

To this end, let $\pi : N \rtimes_{\text{alg}} G \to B(H)$ be a representation. Then we can form $\tilde{H} = H \bar{\otimes} \ell^2(G)$, which is a left $N \rtimes_{\text{alg}} G$-module via the representation $\pi \otimes \lambda$. The only natural right module structure, however, is the $\mathbb{C}G$-module structure given by $1 \otimes \rho$. These structures both extend by continuity to M- and LG-module structures respectively, so that $\tilde{H}$ is an $M \otimes LG^o$-module.

Let us now apply this to the two representations that turn up in the definition above. These are first the standard representation of $N \rtimes_{\text{alg}} G$ on $L^2(M) = L^2(N) \bar{\otimes} \ell^2(G)$, and secondly the representation of $N \rtimes_{\text{alg}} G$ on $L^2(N)$ given by

$$\begin{aligned} \pi(x)y &= xy, \\ \pi(g)y &= g(y) \end{aligned}$$

for $x, y \in N$, $g \in G$. (This is in [22] expressed by letting $N \rtimes_{\text{alg}} G$ act on the tensor product $N \rtimes_{\text{alg}} G \otimes_{\mathbb{C}G} \mathbb{C} \simeq N$. Extending this action to $L^2(N)$ gives us the action above.)

In the first case, we get that

$$\tilde{H} \simeq L^2(N) \otimes \ell^2(G) \otimes \ell^2(G),$$

with the left and right actions given by

$$x(\xi \otimes \delta_g \otimes \delta_h) = g^{-1}(x)\xi \otimes \delta_g \otimes \delta_h$$
$$f(\xi \otimes \delta_g \otimes \delta_h) = \xi \otimes \delta_{fg} \otimes \delta_{fh}$$
$$(\xi \otimes \delta_g \otimes \delta_h)f = \xi \otimes \delta_g \otimes \delta_{hf}$$

for $x \in N$, $\xi \in L^2(N)$ and $f, g, h \in G$. As before, we can untangle the representations through an automorphism ϕ, which now is given by

$$\pi(\xi \otimes \delta_g \otimes \delta_h) = \xi \otimes \delta_g \otimes \delta_{g^{-1}h}.$$

After this untanglement, the left and right actions on $\tilde{H}$ are given by

$$x(\xi \otimes \delta_g \otimes \delta_h) = g^{-1}(x)\xi \otimes \delta_g \otimes \delta_h$$
$$f(\xi \otimes \delta_g \otimes \delta_h) = \xi \otimes \delta_{fg} \otimes \delta_h$$
$$(\xi \otimes \delta_g \otimes \delta_h)f = \xi \otimes \delta_g \otimes \delta_{hf}$$

which, after extending the representations to the generated von Neumann algebras, are the representations of M and LG on

$$\tilde{H} \simeq L^2(M)\bar{\otimes}\ell^2(G)$$

by left and right multiplication, respectively.

For the second representation of $N \rtimes_{\mathrm{alg}} G$ on $L^2(N)$, we get the Hilbert space $\tilde{H} = L^2(N)\bar{\otimes}\ell^2(G)$, with left and right representations given by

$$x(y \otimes \delta_g) = xy \otimes \delta_g,$$
$$f(y \otimes \delta_g) = f(y) \otimes \delta_{fg},$$
$$(y \otimes \delta_g) = y \otimes \delta_{gf},$$

where $x, y \in N$ and $f, g \in G$. If we identify $L^2(N)\bar{\otimes}\ell^2(G)$ with $L^2(M)$, these are exactly the actions on $L^2(M)$ induced by the left and right multiplication. Hence we get representations of M and LG on $L^2(M)$ by left and right multiplication.

By the same argument as before, we see that the process gives rise to a definition of relative L^2-Betti numbers for von Neumann algebras, given by

$$\dim_{M\bar{\otimes}LG^o} \mathrm{Tor}_n^{M\bar{\otimes}LG^o}(M\bar{\otimes}LG^o, M).$$

We may once again generalize this.

Definition 4.3. Let $B \subset A$ be algebras, with B containing the unit of A, and let τ be a finite positive faithful tracial state on A (and hence also on B). Let M and N be the von Neumann algebras generated by A and B respectively in the GNS-representation associated to τ. We then define the *relative L^2-Betti numbers of A with respect to B* to be

$$\beta_n^{(2)}(A, B) = \dim_{M\bar{\otimes}N^o} \mathrm{Tor}_n^{A\otimes B^o}(M\bar{\otimes}N^o, A).$$

However, this definitions turns out to be equivalent to the original one.

Proposition 4.4. *Let $B \subset A$ be algebras with B containing the unit of A, and let τ be a finite positive faithful tracial state on A. If A is flat as right a B-module, then*

$$\beta_k^{(2)}(A, B) = \beta_k^{(2)}(B).$$

In particular, if $N \subset M$ are von Neumann algebras with the same unit, then

$$\beta_k^{(2)}(M, N) = \beta_k^{(2)}(N).$$

Proof. Let M and N be the von Neumann algebras generated by A and B, and consider the functors $(M \bar{\otimes} N^o) \otimes_{N \bar{\otimes} N^o} -$ and $(A \otimes B^o) \otimes_{B \otimes B^o} -$. We claim that they are exact.

For the first functor, this says that $M \bar{\otimes} N^o$ is a flat $N \bar{\otimes} N^o$-module, but these are von Neumann algebras, and so this is our earlier flatness result for von Neumann algebras.

For the second functor, this follows from the flatness of A over B. Indeed, let

$$0 \longrightarrow J \longrightarrow K \longrightarrow L \longrightarrow 0$$

be an exact sequence of $B \otimes B^o$-modules. If we consider these to be B-bimodules, then applying the functor gives us the sequence

$$0 \longrightarrow A \otimes_B J \longrightarrow A \otimes_B K \longrightarrow A \otimes_B L \longrightarrow 0$$

which is exact since A is a flat B-module.

Now all that remains is a computation:

$$\begin{aligned}
\beta_k^{(2)}(B) &= \dim_{N \bar{\otimes} N^o} \operatorname{Tor}_k^{B \otimes B^o}(N \bar{\otimes} N^o, B) \\
&= \dim_{M \bar{\otimes} N^o} (M \bar{\otimes} N^o) \otimes_{N \bar{\otimes} N^o} \operatorname{Tor}_k^{B \otimes B^o}(N \bar{\otimes} N^o, B) \\
&= \dim_{M \bar{\otimes} N^o} \operatorname{Tor}_k^{B \otimes B^o}(M \bar{\otimes} N^o, B) \\
&= \dim_{M \bar{\otimes} N^o} \operatorname{Tor}_k^{A \otimes B^o}(M \bar{\otimes} N^o, A) \\
&= \beta_k^{(2)}(A, B),
\end{aligned}$$

where the second equality follows from the behavior of dimension under induction, the third as homology commutes with exact functors, and the fourth is nothing but a flat base change for Tor. $\square$

Remark. This result can be seen as a kind of excision theorem, reminiscent of the result of Suslin and Wodzicki [18] that for a unital C^*-algebra A and a two-sided ideal I of A we have $K_i(A, I) = K_i(I)$ where the K_i are algebraic K-groups.

It is worthwhile to note that the steps in the above computation correspond exactly to the steps used by Thom to prove the equality of his definition to that of Lück.

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